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Chapter 9

Some Types of HyperNeutrosophic Set (7): Type-m, Nonstationary, Subset-valued, and Complex Refined

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Abstract

This paper builds upon the foundational advancements introduced in [26,39–43]. The Neutrosophic Set provides a versatile mathematical framework for addressing uncertainty through its three membership functions: truth, indeterminacy, and falsity [84]. Extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set have been recently proposed to address increasingly complex and multidimensional problems. Detailed formal definitions of these concepts can be found in [33].

In this paper, we extend the Type-*m*, Nonstationary, Subset-Valued, and Complex Refined Neutrosophic Sets using the Hyperneutrosophic Set and the SuperHyperneutrosophic Set frameworks.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. The analysis utilizes classical set-theoretic operations and extends them into advanced frameworks. For readers seeking a deeper understanding of foundational set theory, resources such as [22, 56, 59, 65] are recommended. Additionally, the referenced literature offers a comprehensive exploration of the principles and applications of Neutrosophic Sets.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To address uncertainty, vagueness, and imprecision in decision-making processes, numerous set-theoretic frameworks have been developed. These frameworks include Fuzzy Sets, which were introduced in foundational works such as those by Zadeh [96–104]. Another prominent framework is Intuitionistic Fuzzy Sets, extensively studied by Atanassov and others [5–10]. Vague Sets, introduced and developed by researchers, also contribute significantly to this domain [3,11,52,68,79]. Furthermore, the Hyperfuzzy Set is known as one of the extended concepts of the Fuzzy Set [13, 25, 35, 36, 54, 60–62, 67, 73, 90].

Neutrosophic Sets, first introduced by Smarandache, offer a powerful means of capturing indeterminacy, allowing for more nuanced decision-making models [28, 29, 31, 38, 45–50, 83, 84, 89]. Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [81–84]. This enhancement allows for a richer and more precise representation of uncertainty and ambiguity.

To address increasingly complex scenarios, HyperNeutrosophic Sets and *n*-SuperHyperNeutrosophic Sets have been developed. These advanced models are particularly suited for high-dimensional and intricate problem spaces [27, 33]. Relevant definitions and simple examples are provided below.

Definition 1.1 (Base Set)**.** A *base set S* is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [32, 78] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [23, 32, 34, 80, 87])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set)**.** [83, 84] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.5 (Neutrosophic Set)**.** *Scenario:* Assessing public opinion on a controversial policy.

Example: Let $X = \{\text{Alice, Bob, Charlie}\}$, representing individuals with varying opinions on the policy. The membership functions represent their support (T), uncertainty (I), and opposition (F) as follows:

- For Alice: $T_A(\text{Alice}) = 0.8$ (80% support), $I_A(\text{Alice}) = 0.1$ (10% uncertain), $F_A(\text{Alice}) = 0.1$ (10% oppose).
- For Bob: $T_A(\text{Bob}) = 0.5$, $I_A(\text{Bob}) = 0.3$, $F_A(\text{Bob}) = 0.2$.
- For Charlie: $T_A(\text{Charlie}) = 0.3$, $I_A(\text{Charlie}) = 0.4$, $F_A(\text{Charlie}) = 0.3$.

This representation allows nuanced analysis, reflecting both certainty and uncertainty in opinions.

Definition 1.6 (HyperNeutrosophic Set)**.** (cf. [24, 27, 33, 35, 85]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0, 1]^3),$$

where $\mathcal{P}([0, 1]^3)$ is the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.7 (HyperNeutrosophic Set)**.** *Scenario:* Analyzing customer satisfaction for multiple products, considering evaluations from different dimensions or individuals.

Example: Let $X = \{\text{Product A, Product B}\}$, where each product has multi-dimensional satisfaction scores represented by sets of neutrosophic triplets:

- For Product A:
$$\tilde{\mu}(\text{Product A}) = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1)\},$$
representing two customers' evaluations where each triplet denotes degrees of truth, indeterminacy, and falsity.
- For Product B:
$$\tilde{\mu}(\text{Product B}) = \{(0.6, 0.3, 0.1), (0.5, 0.4, 0.1)\}.$$

This structure enables richer analysis by aggregating diverse customer feedback for a comprehensive view.

Definition 1.8 (n-SuperHyperNeutrosophic Set)**.** (cf. [24, 27, 33, 35]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 1.9 (n-SuperHyperNeutrosophic Set)**.** *Scenario:* Multi-level hierarchical analysis of climate change impacts.

Example: Let $X = \{\text{Temperature, Rainfall, Sea Level}\}$, representing key factors influenced by climate change. We consider a three-level hierarchy:

- *Level 1:* Regions {Region 1, Region 2}.
- *Level 2:* Countries within regions, e.g., {Country A, Country B, Country C}.
- *Level 3:* Cities within countries, e.g., {City X, City Y, City Z}.

For each level, the n-SuperHyperNeutrosophic Set assigns a family of subsets with membership triplets. For instance:

$$\tilde{A}_3(\text{City X}) = \{(0.8, 0.15, 0.05), (0.7, 0.2, 0.1)\},$$

where each triplet represents truth (T), indeterminacy (I), and falsity (F) degrees at the city level. This approach integrates uncertainty at regional, country, and city scales for holistic decision-making.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Nonstationary Neutrosophic Set

A Nonstationary Neutrosophic Set extends the neutrosophic set by introducing time-dependent truth, indeterminacy, and falsity memberships, which dynamically evolve over a time domain T. Related concepts, such as Nonstationary Fuzzy Sets [4, 4, 51, 57, 58, 95], are also well-known in the literature.

Definition 2.1 (Nonstationary Neutrosophic Set)**.** [37] Let X be a non-empty set, and let T be a time domain (which may be continuous or discrete). A *nonstationary neutrosophic set* $\dot{A}$ on X is defined by three *time-dependent* membership functions:

$$T_{\dot{A}} : T \times X \;\to\; [0,1], \quad I_{\dot{A}} : T \times X \;\to\; [0,1], \quad F_{\dot{A}} : T \times X \;\to\; [0,1],$$

where for each $(t,x) \in T \times X$, the values $T_{\dot{A}}(t,x)$, $I_{\dot{A}}(t,x)$, and $F_{\dot{A}}(t,x)$ represent the *truth*, *indeterminacy*, and *falsity* degrees of x in $\dot{A}$ at time t. These satisfy

$$0 \;\le\; T_{\dot{A}}(t,x) \;+\; I_{\dot{A}}(t,x) \;+\; F_{\dot{A}}(t,x) \;\le\; 3,$$

for all $(t,x) \in T \times X$.

Analogous to the nonstationary fuzzy set, each component can be viewed as a *time-varying perturbation* of the corresponding membership function in a *stationary* neutrosophic set. Specifically, if T_A, I_A, and F_A define a classical neutrosophic set A on X (with no time dependence), then we introduce a set of time-dependent parameters

$$\{p_{T,i}(t),\; p_{I,j}(t),\; p_{F,k}(t)\} \quad \text{for } i,j,k \in \mathcal{I},$$

and define

$$\begin{aligned} T_{\dot{A}}(t,x) &= T_A\big(x;\, p_{T,1}(t), \ldots, p_{T,m}(t)\big), \\ I_{\dot{A}}(t,x) &= I_A\big(x;\, p_{I,1}(t), \ldots, p_{I,n}(t)\big), \\ F_{\dot{A}}(t,x) &= F_A\big(x;\, p_{F,1}(t), \ldots, p_{F,p}(t)\big), \end{aligned}$$

where each parameter function $p_{\cdot,\cdot}(t)$ may evolve over time via a perturbation rule, e.g.

$$p_{T,i}(t) = p_{T,i} + k_{T,i} \cdot f_{T,i}(t),$$

and similarly for the indeterminacy and falsity parameters. In integral notation, the nonstationary neutrosophic set $\dot{A}$ can be expressed as

$$\dot{A} \;=\; \int_{t \in T} \int_{x \in X} \big(T_{\dot{A}}(t,x),\, I_{\dot{A}}(t,x),\, F_{\dot{A}}(t,x)\big) / x \,/\, t.$$

Definition 2.2 (Nonstationary HyperNeutrosophic Set)**.** Let X be a non-empty set, and let T be a time domain (continuous or discrete). A *Nonstationary HyperNeutrosophic Set* $\dot{\tilde{A}}$ on X is a mapping

$$\dot{\tilde{\mu}} : T \times X \;\longrightarrow\; \mathcal{P}\big([0,1]^3\big),$$

where for each pair $(t,x) \in T \times X$,

$$\dot{\tilde{\mu}}(t,x) \;\subseteq\; [0,1]^3,$$

and every element $\big(T_{t,x}, I_{t,x}, F_{t,x}\big) \in \dot{\tilde{\mu}}(t,x)$ satisfies

$$0 \;\le\; T_{t,x} \;+\; I_{t,x} \;+\; F_{t,x} \;\le\; 3.$$

Furthermore, each $\dot{\tilde{\mu}}(t,x)$ can *evolve in time*, meaning that for different values of t, the subsets $\dot{\tilde{\mu}}(t,x) \subseteq [0,1]^3$ may vary.

Interpretation.

- This definition *generalizes* the concept of a Nonstationary Neutrosophic Set by allowing each membership evaluation at time t and element x to be not just a single triple $\big(T, I, F\big)$, but a *set* of possible membership triples in $[0,1]^3$.
- It *simultaneously* generalizes the concept of a HyperNeutrosophic Set by introducing a time dimension and making the hyperneutrosophic membership $\dot{\tilde{\mu}}(t,x)$ *time-dependent*.

Theorem 2.3 (Nonstationary HyperNeutrosophic Set Generalizes Both Nonstationary Neutrosophic Set and HyperNeutrosophic Set)**.**

1. *If, for each* (t, x)*, the set* $\dot{\tilde{\mu}}(t, x) \subseteq [0, 1]^3$ *is always a singleton, i.e.*

$$\dot{\tilde{\mu}}(t, x) \;=\; \big\{\big(T_{\dot{A}}(t, x),\, I_{\dot{A}}(t, x),\, F_{\dot{A}}(t, x)\big)\big\},$$

then we recover a Nonstationary Neutrosophic Set*.*

2. *If we* freeze time *to a single value* $t = t_0$*, then* $\dot{\tilde{\mu}}(t_0, x) \subseteq [0, 1]^3$ *describes a standard HyperNeutrosophic Set (with no time dependence).*

Proof. (1) Suppose each $\dot{\tilde{\mu}}(t, x)$ is a singleton set. Then for each $(t, x) \in T \times X$, there is exactly one triple $\big(T_{t,x}, I_{t,x}, F_{t,x}\big)$. This precisely recovers the definition of a Nonstationary Neutrosophic Set.

(2) Conversely, if we consider a time domain consisting of a single time instant t_0, then $\dot{\tilde{\mu}}(t_0, x) \subseteq [0, 1]^3$ for each x. This yields exactly the definition of a HyperNeutrosophic Set $\tilde{A}$ with membership $\tilde{A}(x) = \dot{\tilde{\mu}}(t_0, x)$. Hence the Nonstationary HyperNeutrosophic structure generalizes both concepts. □

Definition 2.4 (Nonstationary n-SuperHyperNeutrosophic Set)**.** Let X be a non-empty set and let T be a time domain (continuous or discrete). A *Nonstationary n-SuperHyperNeutrosophic Set* $\dot{\tilde{A}}_n$ is defined as a mapping

$$\dot{\tilde{A}}_n \;:\; T \times \mathcal{P}_n(X) \;\longrightarrow\; \mathcal{P}_n\big([0, 1]^3\big),$$

subject to:

- $\mathcal{P}_n(X)$ is the n-th powerset of X, recursively constructed via:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{k+1}(X) = \mathcal{P}\Big(\mathcal{P}_k(X)\Big), \quad k \geq 1.$$

- $\mathcal{P}_n([0, 1]^3)$ is the n-th powerset of the unit cube $[0, 1]^3$.
- For each $\big(t, A\big) \in T \times \mathcal{P}_n(X)$,

$$\dot{\tilde{A}}_n\big(t, A\big) \;\subseteq\; \mathcal{P}_n\big([0, 1]^3\big).$$

- All neutrosophic membership triples $\big(T_{(t,A)},\, I_{(t,A)},\, F_{(t,A)}\big) \in [0, 1]^3$ must satisfy

$$0 \;\leq\; T_{(t,A)} \;+\; I_{(t,A)} \;+\; F_{(t,A)} \;\leq\; 3.$$

Moreover, each membership assignment $\dot{\tilde{A}}_n(t, A) \subseteq \mathcal{P}_n([0, 1]^3)$ can vary with time $t \in T$, making the set *nonstationary*.

Theorem 2.5 (Nonstationary n-SuperHyperNeutrosophic Set Generalizes Both Nonstationary HyperNeutrosophic Set and n-SuperHyperNeutrosophic Set)**.**

1. *By restricting* $n = 1$ *in Definition 2.4, we obtain a* Nonstationary HyperNeutrosophic Set*.*
2. *By restricting the time domain* T *to a single value* t_0*, we obtain a standard* $\big(n{-}SuperHyperNeutrosophicSet\big)$ *with no time dependence.*

Proof. (1) If $n = 1$*,* then $\mathcal{P}_1(X) = \mathcal{P}(X)$ and $\mathcal{P}_1([0, 1]^3) = \mathcal{P}([0, 1]^3)$. Thus the mapping

$$\dot{\tilde{A}}_1 : T \times \mathcal{P}(X) \;\longrightarrow\; \mathcal{P}\big([0, 1]^3\big)$$

coincides with the definition of a *Nonstationary HyperNeutrosophic Set* in Definition 2.2.

(2) If we fix a single time instant $t_0 \in T$, then for each $A \in \mathcal{P}_n(X)$, we assign

$$\dot{\tilde{A}}_n(t_0, A) \;\subseteq\; \mathcal{P}_n([0, 1]^3).$$

Hence the membership is no longer time-dependent, becoming

$$\tilde{A}_n(A) \;=\; \dot{\tilde{A}}_n(t_0, A).$$

This is precisely the definition of an $\big(n - SuperHyperNeutrosophicSet\big)$ that maps $\mathcal{P}_n(X)$ into $\mathcal{P}_n([0, 1]^3$. Therefore, the Nonstationary dimension is removed, and we recover the stationary case. □

Theorem 2.6 (Time-Frozen or Hyper-Frozen Slices)**.** *Let $\dot{\tilde{A}}_n$ be a Nonstationary n-SuperHyperNeutrosophic Set.*

1. Time-Frozen Slices: *For each fixed $t \in T$, the structure $\dot{\tilde{A}}_n(t, \cdot)$ yields an ordinary $(n-SuperHyperNeutrosophicSet)$ on $\mathcal{P}_n(X)$.*
2. Hyper-Frozen Slices: *For each fixed sub-collection $\mathcal{C} \subseteq \mathcal{P}_n(X)$ restricted to singletons or first-level sets, we recover a Nonstationary (but not superhyper) membership arrangement, akin to a Nonstationary Neutrosophic Set or Nonstationary HyperNeutrosophic Set.*

Proof. (1) Follows directly from Theorem 2.5 part (2). *(2)* If we restrict the hyper operation such that only single-level or single-subset outputs are allowed, the superhyper approach collapses to a simpler hyper or non-hyper structure, preserving the time dependence. The details align with the usual embeddings from superhyper to hyper or classical sets. □

2.2 Type-m Neutrosophic Set

A Type-m Neutrosophic Set hierarchically represents truth, indeterminacy, and falsity memberships by recursively nesting Type-$(m-1)$ Neutrosophic Sets for $m \geq 1$ [44]. Related concepts include the Type-2 Neutrosophic Set [1, 18, 55, 63], Type-2 Fuzzy Set [12, 19–21, 53, 64, 66, 66, 69, 70, 72, 92–94], Type-3 Fuzzy Set [14–17, 71, 74], and Type-m Fuzzy Set [2, 75–77, 91, 105]. The definition is provided below.

Definition 2.7 (Type-m Neutrosophic Set)**.** [44] Let X be a non-empty universe of discourse, and let $m \geq 1$. A *Type-m Neutrosophic Set (TMS)* $\mathcal{A}$ on X is defined by three membership functions

$$T_{\mathcal{A}},\ I_{\mathcal{A}},\ F_{\mathcal{A}} : X \longrightarrow \mathcal{M}_{m-1}[0,1],$$

where $\mathcal{M}_{m-1}[0,1]$ denotes the set of all *Type-$(m-1)$ Neutrosophic Sets* whose truth, indeterminacy, and falsity memberships lie in $[0,1]$. Formally,

$$\mathcal{A} \;=\; \Big\{ \langle x,\ T_{\mathcal{A}}(x),\ I_{\mathcal{A}}(x),\ F_{\mathcal{A}}(x) \rangle \;\mid\; x \in X \Big\}.$$

Moreover, at any *terminal level* (i.e. when $m = 1$), the membership functions become

$$T_{\mathcal{A}}(x),\ I_{\mathcal{A}}(x),\ F_{\mathcal{A}}(x) \;\in\; [0,1],$$

subject to the constraint

$$0 \;\leq\; T_{\mathcal{A}}(x) \;+\; I_{\mathcal{A}}(x) \;+\; F_{\mathcal{A}}(x) \;\leq\; 3.$$

This recursion implies that each $(T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}})$ at the $(m-1)$-level is itself a Type-$(m-1)$ Neutrosophic Set, until the recursion terminates at $m = 1$.

2.3 Subset-Valued Neutrosophic Set

Subset-Valued Neutrosophic Set (SSVNS) represents truth, indeterminacy, and falsity as subsets of $[0,1]$, satisfying bounded sum constraints [30].

Definition 2.8 (Subset-Valued Neutrosophic Set)**.** [86, 88] Let X be a space of elements, where $x \in X$ represents a generic element. A Subset-Valued Neutrosophic Set (SSVNS) A is characterized by the following membership functions:

- The *truth membership function* $T_A(x)$,
- The *indeterminacy membership function* $I_A(x)$,
- The *falsity membership function* $F_A(x)$,

where each function satisfies the conditions:

$$T_A(x), I_A(x), F_A(x) \subseteq [0,1], \quad \forall x \in X,$$

and

$$0 \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3.$$

Definition 2.9 (Subset-Valued HyperNeutrosophic Set (SV-HNS))**.** Let X be a non-empty set, and denote $\mathcal{P}_{\neq\emptyset}\big([0,1]\big)$ as the set of all non-empty subsets of $[0,1]$. We define

$$\mathcal{P}^3_{\neq\emptyset}([0,1]) \;=\; \Big\{ (T,I,F) \;\Big|\; T,I,F \in \mathcal{P}_{\neq\emptyset}([0,1]) \Big\},$$

the space of *non-empty subset-triples*. A *Subset-Valued HyperNeutrosophic Set (SV-HNS)* $\widetilde{A}$ is a mapping

$$\widetilde{\mu} : X \;\longrightarrow\; \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big) \setminus \{\emptyset\},$$

where for each $x \in X$,

$$\widetilde{\mu}(x) \;\subseteq\; \Big\{ (T,I,F) \;\Big|\; T \cup I \cup F \subseteq [0,1],\; \sup(T) + \sup(I) + \sup(F) \;\leq\; 3 \Big\},$$

and $\widetilde{\mu}(x) \neq \emptyset$. Equivalently, each $\widetilde{\mu}(x)$ is a *set of subset-triples* $\big(T,I,F\big)$, each triple satisfying

- $T, I, F \subseteq [0,1]$ (non-empty),
- $\sup(T) + \sup(I) + \sup(F) \leq 3$.

Remark 2.10.
- This definition *generalizes* the standard hyperneutrosophic notion by replacing each numerical triple $(T,I,F) \in [0,1]^3$ with subset-triples $\big(T,I,F\big)$ with each of $T, I, F \subseteq [0,1]$.
- The output of $\widetilde{\mu}(x)$ is not a single subset-triple; it is a *set* of such subset-triples. This is the essence of the hyper notion: one element in X can be mapped to multiple membership descriptions simultaneously.

Theorem 2.11 (Subset-Valued HyperNeutrosophic Set generalizes Subset-Valued Neutrosophic Set)**.** *If for each $x \in X$, the hyperneutrosophic mapping $\widetilde{\mu}(x)$ contains exactly one subset-triple $\big(T_A(x), I_A(x), F_A(x)\big)$, then we recover the standard* Subset-Valued Neutrosophic Set *from Definition.*

Proof. In Subset-Valued HyperNeutrosophic structure, we allow $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$. If we force $\widetilde{\mu}(x)$ to be a singleton for all x, i.e.

$$\widetilde{\mu}(x) = \big\{ (T_A(x), I_A(x), F_A(x)) \big\},$$

then there is exactly one triple of subsets describing the membership for each x. This precisely recovers the Subset-Valued Neutrosophic Set, where each x is assigned a unique triple $\big(T_A(x), I_A(x), F_A(x)\big)$. Hence, restricting the hyper notion to singletons collapses the mapping to a single triple per x, i.e., the standard subset-valued membership approach. □

Definition 2.12 (Subset-Valued n-SuperHyperNeutrosophic Set (SV-nSHNS))**.** Let X be a non-empty set, and let $\mathcal{P}_1(X) = \mathcal{P}(X)$, $\mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big)$ for $k \geq 1$. Also define a similar iterative process for subset-triples in $[0,1]$. A *Subset-Valued n-SuperHyperNeutrosophic Set (SV-nSHNS)* is a map

$$\widetilde{A}_n :\; \mathcal{P}_n(X) \;\longrightarrow\; \mathcal{P}_n\Big(\mathcal{P}^3_{\neq\emptyset}([0,1])\Big),$$

subject to:

- For each $A \in \mathcal{P}_n(X)$, $\widetilde{A}_n(A) \subseteq \mathcal{P}_n\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big)$,
- Each *element* of $\widetilde{A}_n(A)$ (call it a *hyper-element*) must be a set of subset-triples $\big(T,I,F\big)$ that satisfy

$$T, I, F \subseteq [0,1], \quad T \cup I \cup F \subseteq [0,1], \quad \sup(T) + \sup(I) + \sup(F) \;\leq\; 3.$$

Equivalently, each $\widetilde{A}_n(A)$ is a *(possibly large) set* whose elements are themselves *collections* of subset-based membership triplets, arranged in up to n-level nested hyper-structures.

Remark 2.13.
- This structure is both *subset-valued* (since each membership is a triple of subsets of $[0,1]$) and *n-superhyper* (since we are mapping from $\mathcal{P}_n(X)$ to an n-level powerset of subset-triples).
- It significantly generalizes the standard n-SuperHyperNeutrosophic Set from $\mathcal{P}_n(X)$ to $\mathcal{P}_n([0,1]^3)$.

Theorem 2.14 (Subset-Valued n-SuperHyperNeutrosophic Set generalizes Subset-Valued HyperNeutrosophic Set)**.** *Let $\widetilde{A}_n$ be a Subset-Valued n-SuperHyperNeutrosophic Set (Definition 2.12). If $n = 1$, we recover the* Subset-Valued HyperNeutrosophic Set (SV-HNS) *from Definition 2.9.*

Proof. If $n = 1$, then $\mathcal{P}_1(X) = \mathcal{P}(X)$, and $\mathcal{P}_1\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big) = \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big)$. Thus the mapping

$$\widetilde{A}_1 : \mathcal{P}(X) \longrightarrow \mathcal{P}\Big(\mathcal{P}^3_{\neq\emptyset}([0,1])\Big) \setminus \{\emptyset\}.$$

But $\mathcal{P}(X)$ in the domain is effectively the set X if we interpret each singleton $\{x\} \in \mathcal{P}(X)$ as an element. By identifying each $A \in \mathcal{P}(X)$ with a single point $x \in X$, we replicate the Subset-Valued *HyperNeutrosophic* approach. More rigorously, the standard hyperneutrosophic definition from Definition 2.9 uses a map

$$\widetilde{\mu} : X \to \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big),$$

where each $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$. Setting $n = 1$ identifies each element $x \in X$ with the subset $\{x\} \in \mathcal{P}(X)$. Hence the domain collapses to first-level subsets, and we precisely recover the Subset-Valued HyperNeutrosophic Set structure. □

Theorem 2.15 (Subset-Valued HyperNeutrosophic Set generalizes Subset-Valued Neutrosophic Set)**.** *If each mapping $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$ is restricted to exactly one triple of subsets, we recover the standard Subset-Valued Neutrosophic Set.*

Proof. This was essentially Theorem 2.11, restated. When the hyper mapping is forced to be a singleton for each x, we get a single subset-triple $\big(T_A(x), I_A(x), F_A(x)\big)$. That exactly matches Definition. □

Theorem 2.16 (Subset-Valued n-SuperHyperNeutrosophic Set extends standard n-SuperHyperNeutrosophic Set)**.** *By imposing that each membership triple $(T, I, F) \in [0,1]^3$ be single-valued subsets (i.e., each is a single-point subset), a Subset-Valued n-SuperHyperNeutrosophic Set reduces to an n-SuperHyperNeutrosophic Set as defined in prior works.*

Proof. In a Subset-Valued n-SuperHyperNeutrosophic Set $\widetilde{A}_n$, each membership hyper-value is an element in

$$\mathcal{P}_n\big(\mathcal{P}^*([0,1]^3)\big),$$

where $\mathcal{P}^*([0,1]^3)$ denotes the power set of $[0,1]^3$ excluding the empty set. If we impose the condition that each subset $\{T, I, F\} \subseteq [0,1]$ is a singleton (i.e., for each coordinate T, I, F, the subset contains exactly one element), then any membership triple (T, I, F) effectively becomes $\{(\tau, \iota, \varphi)\} \subseteq [0,1]^3$, where (τ, ι, φ) is a single point in the unit cube.

This restriction collapses the subset-based membership values to single-point numerical values. Consequently, the mapping domain $\mathcal{P}_n(X)$ to $\mathcal{P}_n([0,1]^3)$ of the Subset-Valued n-SuperHyperNeutrosophic Set coincides with the mapping used in a standard n-SuperHyperNeutrosophic Set. Thus, the subset-based approach is a generalization of the numerical-based approach. □

2.4 Single-Valued Complex Refined Neutrosophic Set

Single-Valued Complex Refined Neutrosophic Set (SVCRNS) refines truth, indeterminacy, and falsity into sub-functions with real (amplitude) and imaginary (phase) components [86]

Definition 2.17 (Single-Valued Complex Refined Neutrosophic Set (SVCRNS))**.** [86] Let X be a space of elements, with $x \in X$ representing a generic element. A *Single-Valued Complex Refined Neutrosophic Set (SVCRNS) A* is defined as follows:

Characteristics

1. *p-Sub-truth Membership Functions:* Each truth membership function is expressed in complex form:
$$T_{A1}(x)e^{iT^I_{A1}(x)}, T_{A2}(x)e^{iT^I_{A2}(x)}, \ldots, T_{Ap}(x)e^{iT^I_{Ap}(x)},$$
where $T_{Ai}(x) \in [0,1]$ represents the real part (amplitude), and $T^I_{Ai}(x) \in [0, 2\pi]$ represents the imaginary part (phase).

2. *r-Sub-indeterminacy Membership Functions:* Each indeterminacy membership function is expressed in complex form:
$$I_{A1}(x)e^{iI^I_{A1}(x)}, I_{A2}(x)e^{iI^I_{A2}(x)}, \ldots, I_{Ar}(x)e^{iI^I_{Ar}(x)},$$
where $I_{Aj}(x) \in [0,1]$ and $I^I_{Aj}(x) \in [0, 2\pi]$.

3. *s-Sub-falsity Membership Functions:* Each falsity membership function is expressed in complex form:
$$F_{A1}(x)e^{iF^I_{A1}(x)}, F_{A2}(x)e^{iF^I_{A2}(x)}, \ldots, F_{As}(x)e^{iF^I_{As}(x)},$$
where $F_{Ak}(x) \in [0,1]$ and $F^I_{Ak}(x) \in [0, 2\pi]$.

Conditions

1. *Real Part Condition (Amplitude):* For all $x \in X$, the real parts satisfy:
$$0 \leq \sum_{i=1}^{p} T_{Ai}(x) + \sum_{j=1}^{r} I_{Aj}(x) + \sum_{k=1}^{s} F_{Ak}(x) \leq p + r + s.$$

2. *Imaginary Part Condition (Phase):* For all $x \in X$, the imaginary parts satisfy:
$$0 \leq \sum_{i=1}^{p} T^I_{Ai}(x) + \sum_{j=1}^{r} I^I_{Aj}(x) + \sum_{k=1}^{s} F^I_{Ak}(x) \leq 2\pi(p + r + s).$$

3. *Parameter Constraints:* The integers $p, r, s \geq 0$, and at least one of p, r, s satisfies ≥ 2:
$$\max(p, r, s) \geq 2.$$

Explanation

- The real part (or amplitude) $T_{Ai}(x), I_{Aj}(x), F_{Ak}(x)$ represents the intensity or degree of truth, indeterminacy, and falsity for each sub-membership function.
- The imaginary part (or phase) $T^I_{Ai}(x), I^I_{Aj}(x), F^I_{Ak}(x)$ introduces a cyclic or periodic component, extending the membership functions into the complex domain.
- The refinement into $p+r+s$ sub-functions allows for a more detailed and multidimensional representation of truth, indeterminacy, and falsity.

Definition 2.18 (Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS))**.** Let X be a non-empty set. Denote by $C_{p,r,s}$ the set of all possible refined complex membership tuples
$$\left(T_1 e^{i\,T^I_1}, \ldots, T_p e^{i\,T^I_p} \;\middle|\; I_1 e^{i\,I^I_1}, \ldots, I_r e^{i\,I^I_r} \;\middle|\; F_1 e^{i\,F^I_1}, \ldots, F_s e^{i\,F^I_s}\right)$$
satisfying the amplitude-phase constraints in Definition. A *Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS)* $\widetilde{H}$ is a mapping
$$\widetilde{H} : X \longrightarrow \mathcal{P}\big(C_{p,r,s}\big) \setminus \{\varnothing\},$$
where for each $x \in X$,
$$\widetilde{H}(x) \subseteq \left\{\big(T_{A1}(x)e^{iT^I_{A1}(x)}, \ldots, F_{As}(x)e^{iF^I_{As}(x)}\big) : \text{satisfying refined constraints}\right\},$$
and $\widetilde{H}(x) \neq \varnothing$. In other words, $\widetilde{H}(x)$ is a (possibly large) set of complex refined membership tuples, each composed of p-sub-truth values, r-sub-indeterminacy values, and s-sub-falsity values in the complex plane, subject to amplitude ≤ 1 and phase $\leq 2\pi$.

Theorem 2.19 (SVCR-HNS generalizes SVCRNS)**.** *If for each $x \in X$, $\widetilde{H}(x)$ is restricted to exactly one refined complex membership tuple $\left\{ (T_{A1}(x)e^{iT^I_{A1}(x)}, \dots) \right\}$, then the SVCR-HNS reduces to a Single-Valued Complex Refined Neutrosophic Set from Definition.*

Proof. By definition, an SVCR-HNS $\widetilde{H}(x)$ is a non-empty set of possible refined membership tuples. If we force $\widetilde{H}(x)$ to be a singleton for each $x \in X$, then each element has exactly one (p, r, s)-refined membership tuple, which is precisely the single-valued mapping in the original sense. The amplitude-phase constraints remain the same, so we get back to a single, refined complex membership assignment. □

Definition 2.20 (Single-Valued Complex Refined n-SuperHyperNeutrosophic Set (SVCR-nSHNS))**.** Let X be a non-empty set, and define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big) \quad \text{for } k \geq 1.$$

Similarly, define a space of refined complex membership structures:

$$C_{p,r,s},$$

and iteratively:

$$\mathcal{P}_n\big(C_{p,r,s}\big) = \underbrace{\mathcal{P}\big(\cdots\mathcal{P}\big)}_{n \text{ times}}\big(C_{p,r,s}\big)\big)\cdots\big).$$

A *Single-Valued Complex Refined **n**-SuperHyperNeutrosophic Set (SVCR-**n**SHNS)* is defined as a mapping:

$$\widetilde{H}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big(C_{p,r,s}\big) \setminus \{\varnothing\},$$

where each image $\widetilde{H}_n(A)$ (for $A \in \mathcal{P}_n(X)$) is a non-empty subset of $C_{p,r,s}$ arranged in up to n-level nested hyper-structures. Each membership tuple within $\widetilde{H}_n(A)$ must satisfy the following conditions:

$$\begin{aligned}
&T_{Ai}(A), I_{Aj}(A), F_{Ak}(A) \in [0,1],\\
&T^I_{Ai}(A), I^I_{Aj}(A), F^I_{Ak}(A) \in [0, 2\pi],\\
&0 \;\leq\; \sum_{i=1}^{p} T_{Ai}(A) + \sum_{j=1}^{r} I_{Aj}(A) + \sum_{k=1}^{s} F_{Ak}(A) \;\leq\; p + r + s,\\
&0 \;\leq\; \sum_{i=1}^{p} T^I_{Ai}(A) + \sum_{j=1}^{r} I^I_{Aj}(A) + \sum_{k=1}^{s} F^I_{Ak}(A) \;\leq\; 2\pi\,(p + r + s).
\end{aligned}$$

Hence, the domain is $\mathcal{P}_n(X)$, and the codomain is $\mathcal{P}_n\big(C_{p,r,s}\big)$.

Theorem 2.21 (SVCR-nSHNS generalizes SVCR-HNS)**.** *If $n = 1$ in Definition 2.20, we recover the Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS) from Definition 2.18.*

Proof. By setting $n = 1$, we have $\mathcal{P}_1(X) = \mathcal{P}(X)$ and $\mathcal{P}_1\big(C_{p,r,s}\big) = \mathcal{P}\big(C_{p,r,s}\big)$. Thus the map

$$\widetilde{H}_1 : \mathcal{P}(X) \longrightarrow \mathcal{P}\big(C_{p,r,s}\big) \setminus \{\varnothing\}$$

becomes precisely the SVCR-HNS definition, except each element of $\mathcal{P}(X)$ is identified with a single point $x \in X$. Indeed, a standard identification is that each $x \in X$ is in one-to-one correspondence with $\{x\} \in \mathcal{P}(X)$. Thus the domain effectively reduces to X in the sense of hyperneutrosophic structures, and we get the SVCR-HNS. □

Theorem 2.22 (SVCR-HNS generalizes SVCRNS)**.** *If each $\widetilde{H}(x)$ in an SVCR-HNS is restricted to a singleton refined complex membership tuple, then we recover the Single-Valued Complex Refined Neutrosophic Set (SVCRNS) from Definition.*

Proof. This was established in Theorem 2.19 but restated here for completeness. Imposing a single membership tuple per element collapses the hyper-based approach to the single-valued approach. □

Theorem 2.23 (SVCR-nSHNS generalizes all simpler forms)**.** *An SVCR-nSuperHyperNeutrosophic Set can be specialized to any of:*

1. *An SVCR-HNS (by setting* $n = 1$*),*
2. *An SVCRNS (by setting* $n = 1$ *and forcing singletons in the hyper sense),*
3. *A standard n-SuperHyperNeutrosophic set (by ignoring the complex and refined sub-components, i.e. setting* $p = r = s = 1$ *with zero phase).*

Proof. (1) $\boldsymbol{n = 1}$ *yields SVCR-HNS:* Directly from Definition 2.20, $\mathcal{P}_1(X) = \mathcal{P}(X)$ and the codomain is $\mathcal{P}_1(C_{p,r,s})$. This matches the hyper definition for single-layer. *(2) Singleton restriction yields SVCRNS:* If each set in the codomain has exactly one refined complex membership tuple, then the hyper notion collapses, leaving a single complex refined membership triple per element (Theorem 2.22). *(3) Setting* $\boldsymbol{p = r = s = 1}$ *and phases to zero yields the standard real-valued superhyperneutrosophic approach:* If each amplitude $= T, I, F \in [0, 1]$ and each phase $\equiv 0$, then each sub-truth membership is just a single real number in $[0, 1]$. Summation constraints revert to the standard T+I+F ≤ 3. The superhyper aspect remains in domain and codomain recursion, but the refined complex perspective is gone. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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